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Buchsbaum Liaison Classes

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INTRODUCTION

This paper is a continuation of the work begun in [BM], studying the (arithmetically) Buchsbaum curves in $\mathbb{P}^3$ from the point of view of liaison. These are curves whose associated Harthshorne–Rao modules have only trivial $k[X_0, ..., X_3]$-multiplication, and are a generalization of arithmetically Cohen–Macaulay curves. In this paper we have two main objectives and an application.

First, we give a fairly detailed description of the kinds of curves that can occur as Buchsbaum curves, above all for those whose associated Hartshorne–Rao modules have diameter two. In the latter case our results are detailed as they were in [BM], where we studied the case of diameter one. This is significant because in [BM] we made heavy use of the work of Lazarsfeld and Rao [LR] to describe the curves, while in the present paper we show that for diameter two or more [LR] cannot be applied and so new techniques had to be developed. We also show that under certain circumstances (in terms of the diameter and shift of the Hartshorne–Rao module and in terms of the smallest surface containing the curve) a Buchsbaum curve cannot be reduced and irreducible. This work is contained in Sections 2 and 3.

Our second main objective is classify the smooth maximal rank Buchsbaum curves in terms of the numerical character, for the diameter two case. Since it is shown in [GM] that maximal rank Buchsbaum curves

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must have Hartshorne–Rao modules of diameter two or less, this work taken together with [BM] completes the classification of smooth maximal rank Buchsbaum curves in terms of the numerical character. This is done in Section 4.

Finally, in Section 5 we use the results about the absence of reduced irreducible curves to give an example of a family of smooth curves which specialize to a singular one in such a way that the dimensions of the components of the Hartshorne–Rao module stay fixed but the module structure changes.

1. Preliminaries

Let k be an algebraically closed field and $\mathbb{P}^3 = \mathbb{P}^3_k$. A *curve* shall mean a one-dimensional subscheme of $\mathbb{P}^3$ which is locally Cohen–Macaulay and equidimensional. If C is a curve then

$$M(C) = \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathscr{I}_C(n))$$

is the *Hartshorne–Rao* module of C and $I(C)$ is its homogeneous ideal. We refer to [RAO, SCH] for the basic definitions and results concerning liaison, and in this section we first collect the notations and more recent results which we shall need, and we recall the technique of Liaison Addition introduced in [SCH].

Recall that a curve C is *aritmetically Buchsbaum* if $M(C)$ has only trivial $k[X_0, ..., X_3]$-multiplication (cf., for example, [BSV, GMV]). This property is clearly preserved under liaison, so we shall call the liaison class of such a curve a *Buchsbaum liaison class.* Observe that the even Buchsbaum liaison classes are parameterized by the set of finite sequences $(n_1, ..., n_t)$ of non-negative integers, where the first and the last are non-zero (representing the dimensions of the components of the "associated module," ignoring the shift).

Definition 1.1. $L_{n_1 \cdots n_t}$ is the even Buchsbaum liaison class corresponding to the sequence $(n_1, ..., n_t)$. If M is the associated module, then $\operatorname{diam} M = t$.

The case $\operatorname{diam} M = 1$ was studied in [BM], and we denote such an even liaison class (in this case also odd) simply by $\mathbf{L}_n$. As we shall see (e.g., Corollary 2.6), this case is very special.

An important result concerning $\mathbf{L}_{n_1 \cdots n_t}$ is the following:

Theorem 1.2 [AM]. *Let $C \in \mathbf{L}_{n_1 \cdots n_t}$ and let $N = n_1 + \cdots + n_t$. Let $\alpha(C) = \min\{\deg F \mid F \in I(C)\}$. Then $\alpha(C) \geqslant 2N$.*

In any even liaison class with associated module M, it is not hard to show that a sufficiently large leftward shift of M cannot be the Hartshorne Rao module of any curve (cf. [SCH] or [M1]. Also, once a curve exists for any particular shift of M then a curve exists for any rightward shift (cf. [SCH]). It is thus of interest to identify the leftmost shift of M for which a curve exists, and to describe the special properties of curves which achieve this shift. (For certain special even liaison classes, some very striking results along these lines are obtained in [LR].)

THEOREM 1.3 [GM]. *Let $C \in \mathbf{L}_{n_1 \cdots n_t}$ and let $N = n_1 + \cdots + n_t$. Then $M(C)_d = 0$ for $d \leqslant 2N - 3$.*

This result says that the Hartshorne–Rao module $M(C)$ for any curve $C \in \mathbf{L}_{n_1 \cdots n_t}$ must begin in degree $2N-2$ or greater, so this gives a lower bound for the leftmost shift of the associated module M for which a curve can exist. We shall show in Theorem 2.1 that this bound is sharp. Then

DEFINITION 1.4. $\mathbf{L}^h_{n_1 \cdots n_t}$ is the set of curve $C \in \mathbf{L}_{n_1 \cdots n_t}$ for which $\dim M(C)_{2N-2+h} = n_1$ and $\dim M(C)_t = 0$ for $t \leqslant (2N-2+h) - 1$.

Thus (assuming Theorem 2.1) $h \geqslant 0$ measures the degree to which $M(C)$ fails to have the leftmost shift, and $\mathbf{L}^0_{n_1 \cdots n_t}$ is the set of curves in the leftmost shift.

Recall that a curve $C \subset \mathbb{P}^3$ is said to have *maximal rank* if, for every integer n, either $I(C)_n = 0$ or $M(C)_n = 0$ (or both). (This is one of several equivalent formulations cf., [BE, BO2].)

THEOREM 1.5. [BM]. (a) *If $C \in \mathbf{L}^0_n$ then* $\deg C = 2n^2$.

(b) *If $C \in \mathbf{L}^h_n$ then* $\deg C \geqslant 2n^2 + 2nh$. *This is obtained if and only if C lies on a surface of degree $2n$; there exist smooth curves of this degree.*

(c) *If $C \in \mathbf{L}^h_n$ has maximal rank then* $\deg C \geqslant 2n^2 + 2nh + \binom{h}{2}$. *There exist smooth maximal rank curves of this degree.*

A fundamental construction in the theory of liaison of curves in $\mathbb{P}^3$ is the technique of "Liaison Addition":

THEOREM 1.6 [SCH]. *Let C, C' be any two curves in $\mathbb{P}^3$ with total ideals I, I'. Let $F \in I$, $F' \in I'$ such that (F, F') is a complete intersection. Then the ideal $FI' + F'I$ defines a closed subscheme Y with*

$$M(Y) \cong M(C)(-\deg F') \oplus M(C')(-\deg F).$$

If C and C' are locally Cohen–Macaulay and equidimensional then so is Y. As sets, $Y = C \cup C' \cup (F \cap F')$.

It is often the case for a given curve that we would like to perform a linkage which is "as small as possible." In this paper we shall use the following result:

THEOREM 1.7 [GM, Remark 3.15.1]. *Let C be a curve in $\mathbb{P}^3$. If $h^2(\mathcal{I}_C(\lambda-2))=0$ and $h^1(\mathcal{I}_C(\nu))=0$ for all $\nu \geq \lambda-1$ then $I(C)$ is generated in degree $\leq \lambda$.*

In particular, this guarantees a linkage using surfaces of degree $\alpha(C)$ and λ.

REMARK/DEFINITION 1.8. We now collect the basic definitions and the first properties of the numerical character of a curve. For proofs and further details see [GP, BM].

(a) Let G be a 0-dimensional subscheme of $\mathbb{P}^2$. Then the sheaf $\mathcal{O}_G$ has a resolution

$$0 \to \bigoplus_0^{\sigma-1} \mathcal{O}_{\mathbb{P}^2}(-n_i) \to \bigoplus_0^{\sigma-1} \mathcal{O}_{\mathbb{P}^2}(-i) \to \mathcal{O}_G \to 0,$$

where $n_0 \geq n_1 \geq \cdots \geq n_{\sigma-1} \geq \sigma$ is a sequence of integers which is called the *numerical character* of G, and σ is the smallest degree of a curve in $\mathbb{P}^2$ containing G.

(b) If $Y \subset \mathbb{P}^3$ is a curve and H is a general plane, then the numerical character of $Y \cap H$ is independent of H, and it is called the *numerical character* of Y.

(c) If Y is integral, this sequence is without gaps (i.e., $n_i \leq n_{i+1}+1$).

Remark 1.9 [GP]. If G is a collection of points of $\mathbb{P}_k^2$ with numerical character $N=(n_0, ..., n_{\sigma-1})$, then for any p

$$h^1(\mathbb{P}_k^2, \mathcal{I}_G(p)) = \sum_0^{\sigma-1} (n_i-p-1)_+ - \sum_0^{\sigma-1} (i-p-1)_+,$$

where $(a)_+ = \max\{a, 0\}$. Also,

$$\deg(G) = \sum_0^{\sigma-1} (n_i - i).$$

Notation 1.10. Let $N=(n_0, ..., n_{\sigma-1})$ be a sequence of positive integers with $n_0 \geq n_1 \geq \cdots \geq n_{\sigma-1} \geq \sigma$. Then

$$A_i = \#\{j \mid n_j = i\} \qquad \text{for any } i,$$

$$E = \#\{j \mid n_j = n_0\}.$$

For us, N will always be the numerical character of a curve. If it is not clear from the context to which curve, say Y, we are referring we will use the notation $E(Y)$ and $A_i(Y)$ for E and A_i, respectively.

In [BM] we proved the following classification for certain Buchsbaum classes:

THEOREM 1.11. *Let* $N=(n_0, n_1, ..., n_{\sigma-1})$ *be a sequence of integers without gaps satisfying* $n_0 \geqslant n_1 \geqslant \cdots \geqslant n_{\sigma-1} \geqslant \sigma$

$$\begin{aligned} &\sigma \geqslant 2n-1 \\ &A_\sigma \geqslant n-1 \qquad\qquad [*n] \\ &A_{\sigma+1} \geqslant n, \qquad A_{\sigma+1}=n \Rightarrow A_t=0\ \forall t>\sigma+1. \end{aligned}$$

Then there exists a smooth maximal rank curve $Y \in \mathbf{L}_n$ *such that* $N(Y)=N$.

Conversely, Let $Y \in \mathbf{L}_n$ *be a smooth rank curve. Then its numerical character is a sequence of integers without gaps satisfying* $[*n]$.

In Section 2 of [BM] there is a discussion of the non-maximal rank case and of the singular case.

Since we are interested in maximal rank arithmetically Buchsbaum curves, the following theorem is important here.

THEOREM 1.12 [GM]. *If C is an arithmetically Buchsbaum curve of maximal rank then* $\operatorname{diam} M(C) \leqslant 2$.

It has come to our attention that this result is also proved in [EF].

2. RESULTS ON $L_{n_1 \cdots n_t}$

As mentioned earlier, most of the results in this paper concern the Buchsbaum liaison classes whose associated HR module has diameter 2. However, there are some things we can say about the general case (some of which we shall apply to the diameter 2 case). Consider the Buchsbaum liaison class $\mathbf{L}_{n_1 \cdots n_t}$ $(n_1 \neq 0,\ n_t \neq 0)$. Let $N=\Sigma n_i$ as above.

THEOREM 2.1 (Construction of a Minimal Curve in $\mathbf{L}_{n_1 \cdots n_t}$). *There exists a curve in* $\mathbf{L}_{n_1 \cdots n_t}$ *whose leftmost non-zero component of the HR module occurs in degree* $2N-2$.

Proof. We apply Liaison Addition. The curve that we produce is minimal in the even liaison class with respect to the (leftward) shift of the HR module, by Theorem 1.3. For diameter 1 (cf. [BM]) and diameter 2 (see the next section) this curve is also minimal in the even liaison class with respect to degree. *We conjecture that this is true for all* t.

The curve we produce has degree $2N^2+2n_2+4n_3+\cdots+2(t-1)n_t$. Furthermore, the smallest surface containing this curve has Amasaki's minimial degree $2N$. We proceed by induction on t.

For $t=1$ these curves were produced in [BM]. Now, if $n_2 \neq 0$ assume that we have a curve $Y \in \mathbf{L}_{n_2 \cdots n_t}$ satisfying the above conditions. Let $F \in I(Y)$ have degree $2(n_2+\cdots+n_t)$. Choose a minimal $Z \in \mathbf{L}_{n_1}$ and $G \in I(Z)$ such that $\deg G = 2n_1+1$ and G and F have no common component. Let C be the curve thus obtained by Liaison Addition. Then it is a simple computation to check that C has the desired HR module, degree, and smallest generator for its ideal. (For the latter, take any FH where $H \in I(Z)$ of degree $2n_1$.)

Finally, suppose $n_2=\cdots=n_r=0$ and $n_{r+1}\neq 0$. Again by induction we can find a curve $Y \in \mathbf{L}_{n_{r+1}\cdots n_t}$ satisfying the conditions, and a minimal $Z \in L_{n_1}$. Choose $F \in I(Y)$ of degree $2(n_{r+1}+\cdots+n_t)$ and $G \in I(Z)$ of degree $2n_1+r$ such that F and G have no common component. Then performing Liaison Addition we obtain a curve C, and it is not hard to show that this also has the desired Hartshorne–Rao module, degree, and smallest generator for its ideal. ∎

Remark 2.2. Note that $h^0(\mathscr{I}_C(2N))=h^0(\mathscr{I}_Z(2n_1))=3n_1+1$ (cf. [BM, Proposition 2.6]). We conjecture that all minimal curves have this property. See also Remark 3.9.

A recurring phenomenon which we shall find for Buchsbaum liaison classes (and we conjecture for all liaison classes) is that those curves lying on surfaces of minimal degree (here $2N$) are very special. For example, at least in diameters 1 and 2 it is the case that they are exactly the curves of minimal degree in each shift. The first observations along this line are the following. Recall that for a curve C,

$$e(C)=\max\{n \mid h^1(C,\mathscr{O}_C(n))\neq 0\}=\max\{n \mid h^2(\mathbb{P}^3,\mathscr{I}_C(n))\neq 0\}.$$

(The second equality comes from a standard exact sequence.)

PROPOSITION 2.3. *If* $C \in \mathbf{L}^h_{n_1\cdots n_t}$ *lies on a surface of degree* $2N$ *then* $e(C)\leqslant 2N+h-5+t$.

Proof. Suppose not. Then $h^2(\mathscr{I}_C(2N+h-4+t))\neq 0$ and we have

$\mathscr{I}_C$	$2N-2+h$	$2N-1+h$	$2N+h$	$\cdots$	$2N-4+t+h$	$2N-3+t+h$	$2N-2+t+h$
h^0			*		*	*	*
h^1	n_1	n_2	n_3		n_{t-1}	n_t	0
h^2					*		

Let $e = e(C)$. By Theorem 1.7 we have a link using a surface of degree $2N$ and one of degree $e+3$ to a curve C'. Then by [PS] we have an exact sequence

$$0 \to \mathcal{I}_X(2N-1) \to \mathcal{I}_{C'}(2N-1) \to \omega_{C}(-e) \to 0,$$

where X is the complete intersection. Taking cohomology, note that $h^0(\mathcal{I}_{C'}(2N-1)) = 0$ by Amasaki's bound (Theorem 1.2) and $h^1(\mathcal{I}_X(2N-1)) = 0$ since X is arithmetically Cohen–Macaulay. Thus $h^0(\omega_C(-e)) = 0$, contradicting the definition of e. ∎

COROLLARY 2.4. *If $C \in \mathbf{L}^h_{n_1 \cdots n_t}$ lies on a surface of degree $2N$ then C can be directly linked to a curve in $\mathbf{L}^0_{n_t \cdots n_1}$.*

Proof. Again invoking Theorem 1.7, we have a link using a surface of degree $2N$ and one of degree $2N-1+t+h$. The result then follows from Hartshorne's theorem (cf. [RAO]). ∎

Note that a link using our surface of degree $2N$ and one of degree $< 2N-1+t+h$ is impossible—the residual curve would be shifted too far to the left, violating Theorem 1.3.

COROLLARY 2.5. *If $C \in \mathbf{L}^h_{n_1 \cdots n_t}$ lies on a surface of degree $2N$ then $e(C) = 2N-5+t+h$.*

Proof. We know from Proposition 2.3 that $e(C) \leqslant 2N-5+t+h$. Now perform the link guaranteed by Corollary 2.4 to $C' \in \mathbf{L}^0_{n_t \cdots n_1}$. Again from [PS] we have an exact sequence

$$0 \to \mathcal{I}_X(2N) \to \mathcal{I}_{C'}(2N) \to \omega_C(5-2N-h-t) \to 0.$$

The case $h=0$, $t=1$ follows from [BM, Proposition 2.6]. Otherwise, $2N-1+t+h > 2N$ so $h^0(\mathcal{I}_X(2N)) = 1$. But by [GM, Corollary 4.3], $h^0(\mathcal{I}_{C'}(2N)) \geqslant 2n_t + 1 > 1$. It follows that $h^0(\omega_C(5-2N-h-t)) \neq 0$ and $e(C) = 2N-5+t+h$. ∎

An important question for a liaison class of curve in $\mathbb{P}^3$ is whether or not it satisfies the hypothesis of [LR] that there exists a curve C not lying on any surface of degree $e(C)+3$. If this is satisfied, [LR] provides a powerful tool for studying the liaison class—this was used in [BM] for $\mathbf{L}_n$. We are now able to show that for diameter $\geqslant 2$ this hypothesis is not satisfied by Buchsbaum curves.

COROLLARY 2.6. *If $t \geqslant 2$ then there is no curve in $\mathbf{L}_{n_1 \cdots n_t}$ satisfying the hypothesis of* [LR].

Proof. If such a curve C did exist, it would have to lie in $\mathbf{L}^0_{n_1 \cdots n_t}$ since it is shown in [LR] that such a curve lies in the minimal (i.e., leftmost) shift of the module. Then $\dim M(C)_{2N-2} = n_1 \neq 0$, so by [GM, Corollary 3.9(c)], C lies on a surface of degree $2N$. But then $e(C) = 2N - 5 + t \geqslant 2N - 3$. ∎

In the next section we shall list the possible degrees for curves in each shift, *in diameter two*. Unfortunately, this argument does not seem to extend to the general case. Nevertheless, part of the construction does extend, and from it we at least obtain the following partial results.

PROPOSITION 2.7. *Let* $C \in \mathbf{L}^h_{n_1 \cdots n_t}$, $h > 0$. *Let* $\alpha = \alpha(C)$ *and assume that* $\alpha < 2N + h$, *i.e.,* $M(C)_{\alpha-2} = 0$. *Then C can be linked to a curve* $C' \in \mathbf{L}^{h-1}_{n_1 \cdots n_t}$ *with* $\deg C' < \deg C$.

Proof. We will consider two cases, depending on whether or not C lies on a surface of degree $2N$. In either case, the statement $\deg C' < \deg C$ is a simple computation and is omitted.

Case 1. C lies on a surface of degree $2N$.

Then by Corollary 2.4 we can use a surface of degree $2N$ and one of degree $2N - 1 + t + h$ to link C to a curve $Y \in \mathbf{L}^0_{n_t \cdots n_1}$. Now, Y clearly lies on a surface of degree $2N$, so by Theorem 1.7 we can link Y to $C' \in \mathbf{L}^{h-1}_{n_1 \cdots n_t}$ using a surface of degree $2N$ and one of degree $2N - 1 + t + h - 1 \geqslant 2N - 1 + t$.

Case 2. C does not lie on a surface of degree $2N$.

Let $e = e(C)$. Suppose first that $e \geqslant 2N - 4 + t + h$. We then have

$\mathscr{I}_C$	$2N-2+h$	$2N-1+h$	$2N+h$	$\cdots$	$2N-4+t+h$	$2N-3+t+h$	$2N-2+t+h$
h^0			*	$\cdots$	*	*	*
h^1	n_1	n_2	n_3		n_{t-1}	n_t	0
h^2					*		

and again we can link C to a curve Y using surfaces of degree $\alpha = \alpha(C)$ and $e + 3$. As before, we have an exact sequence

$$0 \to \mathscr{I}_X(\alpha - 1) \to \mathscr{I}_Y(\alpha - 1) \to \omega_C(-e) \to 0$$

so by definition of e, Y lies on a surface of degree $\alpha - 1$. Now, it is not hard to show that a link can then be performed on Y using surfaces of degree $\alpha - 1$ and $e + 3$ to produce C'.

It only remains to check the case $e \leqslant 2N - 5 + t + h$. Now,

$\mathscr{I}_C$	$2N-2+h$	$2N-1+h$	$2N+h$	$\cdots$	$2N-4+t+h$	$2N-3+t+h$	$2N-2+t+h$
h^0			*		*	*	*
h^1	n_1	n_2	n_3		n_{t-1}	n_t	0
h^2					0	0	

This time we can link C to a curve Y using surfaces of degree α and $2N-1+t+h$ and we have an exact sequence

$$0 \to \mathscr{I}_X(\alpha-1) \to \mathscr{I}_C(\alpha-1) \to \omega_Y(4-2N-t-h) \to 0$$

so $h^2(\mathscr{I}_Y(2N-4+t+h))=0$. Also, the rightmost non-zero component of $M(Y)$ occurs in degree $\alpha+t-3$, so $h^1(\mathscr{I}_Y(k))=0$ for $k \geqslant \alpha+t-2$. But $\alpha \leqslant 2N+h-1$ by hypothesis so $\alpha+t-2 \leqslant 2N-3+t+h$. Hence $h^1(\mathscr{I}_Y(2N-3+t+h))=0$ so we can perform a link on Y using surfaces of degree α and $2N-2+t+h$ to produce the desired curve C'. ∎

Remark 2.8. We know from [GM, Corollary 3.9(c)] that $\alpha \leqslant 2N+h$, so this proposition is really only missing the case $\alpha = 2N+h$. The problem with this case is that Theorem 1.7 only lets us link Y using surfaces of degree α and $2N-1+t+h$ again, so there is no improvement. These curves with $M(C)_{\alpha-2} \neq 0$ also distinguished themselves on several occasions in [GM]. In the diameter two case this condition is equivalent to maximal rank, and we shall handle this with the Riemann–Roch theorem in the next section.

We now consider the question of irreducibility, both for curves and for surfaces containing them. Since all Buchsbaum curves except for two skew lines (in $\mathbf{L}_1$) are connected, there results bear on the question of when there can exist a smooth curve. Since most of these results concern $\mathbf{L}^h_{n_1 \cdots n_t}$ for $h \leqslant t-2$, these are really results about the case of diameter $\geqslant 2$. Indeed, many of these statements (e.g., Corollary 2.11) are false in $\mathbf{L}_n$ (cf. [BM]).

LEMMA 2.9. (a) *No curve in $\mathbf{L}^h_{n_1 \cdots n_t}$ which lies on a surface Σ of degree $2N+r$ $(0 \leqslant r \leqslant h)$ can lie on an irreducible surface of degree $\leqslant 2N+t+h-r-2$, except possibly Σ itself.*

(b) *If a curve $C \in \mathbf{L}^h_{n_1 \cdots n_t}$ lies on an irreducible surface Σ of degree $2N+r$ $(0 \leqslant r \leqslant h)$ then every surface in I_C of degree $\leqslant 2N+t+h-r-2$ must have Σ as a component.*

Proof. The point of both (a) and (b) is that if not we could link C via these surfaces to a curve which has a non-zero HR module in degree $\leqslant 2N-3$. ∎

LEMMA 2.10. *Let $C \in \mathbf{L}^h_{n_1 \cdots n_t}$ with $h \leqslant t-2$ and assume that $\alpha = \alpha(C) = 2N+h$ (i.e., $M(C)_{\alpha-2} \neq 0$). Then C cannot lie on an irreducible surface of degree $2N+h$. In particular, C cannot be reduced and irreducible.*

Proof. By [GM, Corollary 4.3], $h^0(\mathscr{I}_C(2N+h)) \geqslant 2n_1+1 > 2$. Hence if C lies on an irreducible surface Σ of degree $2N+h$, we can perform a link using Σ and another surface of degree $2N+h$. But the residual curve Y has its leftmost non-zero component of $M(Y)$ in degree $-(2N+t+h-3)+2N+h+2N+h-4=2N+h-t-1 \leqslant 2N-3$, contradicting [GM, Corollary 3.10(a)]. The second statement follows immediately. Note that the hypothesis $h \leqslant t-2$ makes this lemma vacuous in the diameter one case since h is always $\geqslant 0$. ▮

COROLLARY 2.11. (a) *No curve in $\mathbf{L}_{n_1 \cdots n_t}$ lies on an irreducible surface of degree $2N$ (for $t \geqslant 2$).*

(b) *There is no reduced irreducible curve in $\mathbf{L}_{n_1 \cdots n_t}$ lying on a surface of degree $2N$.*

Proof. For (a), if such a curve C exists then by Corollary 2.4 we can directly link C using this surface to a curve $Y \in \mathbf{L}^0_{n_t \cdots n_1}$, which then also lies on an irreducible surface of degree $2N$. This violates Lemma 2.10 ($h=0$). Part (b) follows from (a) and Amasaki's bound. ▮

THEOREM 2.12. *Let $C \in \mathbf{L}^h_{n_1 \cdots n_t}$ with $h \leqslant t-2$. Then C cannot be reduced and irreducible.*

Proof. We consider two cases.

Case 1: $e(C) \leqslant 2N+t+h-4$.

Suppose C is reduced and irreducible. Say $\alpha(C)=2N+r$. By Lemma 2.10, $r \leqslant h-1$, and since C is irreducible we may assume C lies on an irreducible surface Σ of degree $2N+r$.

Using Theorem 1.7 as above we may link C to a curve Y using surfaces of degree $2N+r$ and $2N+t+h-1$. By Hartshorne's theorem, $Y \in \mathbf{L}^r_{n_t \cdots n_1}$. We have an exact sequence

$$0 \to \mathscr{I}_X(2N+t+h-r-2) \to \mathscr{I}_C(2N+t+h-r-2)$$
$$\to \omega_Y(3-2N-2r) \to 0.$$

By Lemma 2.9(b), the first two terms in the associated long exact sequence have the same dimension. Consequently, $e(Y) \leqslant 2N+2r-4 \leqslant 2N+r+h-5 \leqslant 2N+r+t-7$. The cohomology of $\mathscr{I}_Y$ looks like

$\mathscr{I}_Y$	$2N-2+r$	$2N-1+r$	$2N+r$	$\cdots$	$2N-5+t+r$	$2N-4+t+r$	$2N-3+t+r$	$2N-2+t+r$
h^0			$*$		$*$	$*$	$*$	$*$
h^1	n_t	n_{t-1}	n_{t-2}		n_3	n_2	n_1	0
h^2					0	0	0	0

As usual we can perform a link using our irreducible surface Σ of degree

$2N+r$ and a surface of degree $2N+t+r-1$, to a curve C'. We have an exact sequence

$$0 \to \mathscr{I}_X(2N+r-1) \to \mathscr{I}_{C'}(2N+r-1) \to \omega_Y(4-2N-r-t) \to 0$$

so since the first and third terms of the associated long exact sequence are 0, we have $\alpha(C') = 2N+r$. On the other hand, Hartshorne's theorem gives $C' \in \mathbf{L}^r_{n_1 \cdots n_t}$. Consequently, C' satisfies the hypothesis of Lemma 2.10. But C' lies on Σ. Contradiction.

Case 2. $e(C) \geqslant 2N+t+h-3$.

This case is similar to the first part of Case 2 of Proposition 2.7. Suppose C is reduced and irreducible with $\alpha(C) = 2N+r \leqslant 2N+h-1$ (by Lemma 2.10) and C lies on an irreducible surface Σ of degree $2N+r$. By Theorem 1.7 we have a link using Σ and a surface of degree $e+3$ to a curve Y. As before, we have an exact sequence

$$0 \to \mathscr{I}_X(2N+r-1) \to \mathscr{I}_Y(2N+r-1) \to \omega_C(-e) \to 0$$

from which we get $h^0(\mathscr{I}_Y(2N+r-1)) > 0$. Since Σ is irreducible we then have a link using a surface of degree $2N+r-1$ and one of degree $2N+r$ (namely Σ), to a residual curve C'.

Now, the leftmost non-zero component of $M(C)$ is in degree $2N-2+h$ so the rightmost non-zero component of $M(Y)$ is in degree $r+e-h+1$. Then the leftmost non-zero component of $M(C')$ is in degree

$$\begin{aligned} 4N+r-e+h-6 &\leqslant 4N+r+h-6-(2N+t+h-3) \\ &= 2N+r-t-3 \\ &\leqslant 2N+(h-1)-t-3 \\ &\leqslant 2N+(t-3)-t-3 \\ &= 2N-6. \end{aligned}$$

Contradiction. ∎

3. The Diameter Two Case

In this section we describe the possible degrees that may occur for curves in $\mathbf{L}_{mn}$ in various situations, namely (1) the possible degrees for curves in each shift, (2) the possible degrees for maximal rank curves in each shift, and (3) the possible degrees for reduced irreducible maximal rank curves in each shift. We show that all of the degrees allowed by (1) and (2) can actually arise. One of the consequences of the work in the next section will be that our bounds for (3) are actually achieved by smooth curves.

These results are very similar to the diameter one case (Theorem 1.5) except that the latter were obtained using [LR]. Since we have shown that [LR] cannot be applied in the case of diameter two or more (Corollary 2.6), we develop other techniques to achieve this goal. These techniques should be applicable to non-Buchsbaum situations as well (see Section 6). The main results of this section are summarized in the following theorem. (Compare with Theorem 1.5.) As usual, let $N = m + n$.

THEOREM 3.1. (a) *If* $C \in \mathbf{L}^0_{mn}$ *then* $\deg C = 2N^2 + 2n$.

(b) *If* $C \in \mathbf{L}^h_{mn}$ *then* $\deg C \geq (2N^2 + 2n) + 2Nh$. *This is obtained if and only if C lies on a surface of degree* $2N$ (*and hence such a C is never smooth, by Corollary* 2.11).

(c) *If* $C \in \mathbf{L}^h_{mn}$ *has maximal rank then* $\deg C \geq (2N^2 + 2n) + 2Nh + \binom{h+1}{2}$.

(d) *If* $C \in \mathbf{L}^h_{mn}$ *is a reduced maximal rank curve then*

$$(2N^2 + 2n) + 2Nh + \binom{h+1}{2} \leq \deg C \leq (2N^2 + 2n) + 2Nh + h^2.$$

Note that once we have (a), curves of any degree $\geq$ the bounds in (b) and (c) can be obtained via basic double links (cf. [BM, Remarks 2.3 and 5.2]). It is not hard to check, using the numerical character computations in Section 4, that there exist smooth maximal rank curves of each degree allowed in (d). The key to our ability to make these sharp bounds for $t = 2$ but not for $t \geq 3$ lies in the following lemma. Of course a similar result can be obtained for $t \geq 3$, but it cannot be applied so neatly as will be done now for $t = 2$. (See, for example, Corollary 3.3.)

LEMMA 3.2. *Let* $C \in \mathbf{L}^h_{mn}$. *Then*

$$\begin{aligned} &h^1(\mathcal{O}_C(2N + h - 2)) - h^1(\mathcal{O}_C(2N + h - 1)) \\ &\quad = \deg C - \left\{(2N^2 + 2n) + 2Nh + \binom{h+1}{2} - [h^0(\mathcal{I}_C(2N + h - 1)) \right. \\ &\qquad \left. - h^0(\mathcal{I}_C(2N + h - 2))]\right\}. \end{aligned}$$

Proof. Let p_a be the (arithmetic) genus of C and recall that $\chi = h^0 - h^1$. From the Riemann–Roch theorem we have

$$\chi(\mathcal{O}_C(2N + h - 1)) = (2N + h - 1)(\deg C) - p_a + 1$$

$$\chi(\mathcal{O}_C(2N + h - 2)) = (2N + h - 2)(\deg C) - p_a + 1.$$

From the exact sequence

$$0 \to \mathscr{I}_C \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_C \to 0$$

we have

$$h^0(\mathcal{O}_C(2N+h-1)) = \binom{2N+h+2}{3} + n - h^0(\mathscr{I}_C(2N+h-1))$$

$$h^0(\mathcal{O}_C(2N+h-2)) = \binom{2N+h+1}{3} + m - h^0(\mathscr{I}_C(2N+h-2)).$$

Combining these gives the result (after a few lines of computation). ∎

COROLLARY 3.3. *If $C \in \mathbf{L}^h_{mn}$ has maximal rank then*

$$\deg C \geqslant (2N^2 + 2n) + 2Nh + \binom{h+1}{2}.$$

Proof. We know that $h^1(\mathcal{O}_C(2N+h-2)) - h^1(\mathcal{O}_C(2N+h-1)) \geqslant 0$, and $h^0(\mathscr{I}_C(2N+h-1)) = h^0(\mathscr{I}_C(2N+h-2)) = 0$ by maximal rank. ∎

COROLLARY 3.4. *If $C \in \mathbf{L}^0_{mn}$ then* $\deg C = 2N^2 + 2n$.

Proof. We have $e(C) = 2N-3$ by Corollary 2.5, and C has maximal rank by Amasaki's bound and the definition of $\mathbf{L}^0_{mn}$. Hence applying Lemma 3.2 with $h=0$ gives $0 = \deg C - \{2N^2 + 2n\}$. ∎

COROLLARY 3.5. *Let $C \in \mathbf{L}^h_{mn}$ and assume that C lies on a surface of degree $2N$. Then*

(a) $\deg C = 2N^2 + 2n + 2Nh$.

(b) *If $h \geqslant 1$ then $h^0(\mathscr{I}_C(2N+h-1)) = \binom{h+2}{3}$. In particular, C lies on a unique surface Σ of degree $2N$ and any surface of degree $2N+k$ $(0 \leqslant k \leqslant h-1)$ contains Σ as a component. (Of course, Σ is not irreducible by Corollary* 2.11.)

Proof. By Corollary 2.4, C can be directly linked to $C' \in \mathbf{L}^0_{nm}$ using surfaces of degree $2N$ and $2N+h+1$, and $\deg C' = 2N^2 + 2m$. Then (a) is a simple computation. For (b), consider the exact sequence

$$0 \to \mathscr{I}_X(2N+h-1) \to \mathscr{I}_C(2N+h-1) \to \omega_{C'}(2-2N) \to 0,$$

We know that $e(C') = 2N-3$ so $h^0(\mathscr{I}_C(2N+h-1)) = h^0(\mathscr{I}_X(2N+h-1)) = \binom{h+2}{3}$ (since the second generator of I_X is in degree $2N+h+1$). ∎

PROPOSITION 3.6. *Let $C \in \mathbf{L}_{mn}^h$, $h \geqslant 1$. Assume that C does not lie on a surface of degree $2N$. Then* $\deg C > 2N^2 + 2n + 2Nh$.

Proof. We proceed by induction on h. For $h = 1$, the assumption that C does not lie on a surface of degree $2N$ means that C has maximal rank, so the result follows from Corollary 3.3.

Now assume that $h > 1$ and that the proposition is true for all shifts $< h$. Let $C \in \mathbf{L}_{mn}^h$ and let $\alpha = \alpha(C)$. If C has maximal rank then we are done by Corollary 3.3, so we may assume that C does not have maximal rank. This means that $\alpha \leqslant 2N + h - 1$. The buld of the work, then, is contained in Proposition 2.7—we now merely make some specific computations for this case.

Let $e = e(C)$. If $e \geqslant 2N + h - 2$ then we have a link using surfaces of degree α and $e + 3$ to a curve Y, which in turn is linked by surfaces of degree $\alpha - 1$ and $e + 3$ to $C' \in \mathbf{L}_{mn}^{h-1}$. Now,

$$\begin{aligned}\deg C &= \alpha(e+3) - \deg Y \\ &= \alpha(e+3) - [(\alpha - 1)(e+3) - \deg C'] \\ &= e + 3 + \deg C' \\ &\geqslant 2N + h + 1 + \deg C' \\ &> 2N + \deg C'.\end{aligned}$$

Now, if C' lies on a surface of degree $2N$ then $\deg C' = 2N^2 + 2n + 2N(h-1)$ by Corollary 3.5. If not then $\deg C' > 2N^2 + 2n + 2N(h-1)$ by induction. In either case, $\deg C > 2N^2 + 2n + 2Nh$.

Now suppose that $e(C) \leqslant 2N + h - 3$. We have a link using surfaces of degree α and $2N + h + 1$ to Y and then one using surfaces of degree α and $2N + h$ to $C' \in \mathbf{L}_{mn}^{h-1}$. Computing as above we get

$$\deg C = \alpha + \deg C' > 2N + \deg C'$$

and the same argument finishes the proof. ∎

PROPOSITION 3.7. *Let $C \in \mathbf{L}_{mn}^h$ have maximal rank. If C is reduced and irreducible then*

$$(2N^2 + 2n) + 2Nh + \binom{h+1}{2} \leqslant \deg C \leqslant (2N^2 + 2n) + 2Nh + h^2.$$

Proof. The first inequality is Corollary 3.3. For the second, note that $\alpha = \alpha(C) = 2N + h$ (by the maximal rank assumption and [GM, Corollary 3.9]). Now, by [GM, Corollary 4.3], $h^0(\mathscr{I}_C(2N+h)) \geqslant 2m + 1$.

Since C is reduced and irreducible, we can use two surfaces of degree $2N+h$ to link C to a curve $C' \in \mathbf{L}_{nm}^{h-1}$. Then

$$\begin{aligned}\deg C &= (2N+h)^2 - \deg C' \\ &\leqslant (2N+h)^2 - [2N^2 + 2m + 2N(h-1)] \\ &= 2N^2 + 2n + 2Nh + h^2. \quad \blacksquare\end{aligned}$$

As mentioned earlier, we shall show in the next section that for each degree in this range there actually exists a smooth maximal rank curve of that degree in L_{mn}^h.

COROLLARY 3.8. *Let $C \in \mathbf{L}_{mn}^h$ be a reduced irreducible maximal rank curve and link to $C' \in \mathbf{L}_{nm}^{h-1}$ as above. Then C' has maximal rank if and only if the degree of C is the lower bound of Proposition* 3.7.

Proof. Assume that C' has maximal rank, so $\deg C' \geqslant (2N^2 + 2m) + 2N(h-1) + \binom{h}{2}$. Then substituting this into the equation $\deg C = (2N+h)^2 - \deg C'$ gives $\deg C \leqslant (2N^2 + 2n) + 2Nh + \binom{h+1}{2}$. Since this is also the lower bound, we have equality.

Conversely, assume that $\deg C = (2N^2 + 2n) + 2Nh + \binom{h+1}{2}$. By Lemma 3.2 we have $h^1(\mathcal{O}_C(2N+h-2)) - h^1(\mathcal{O}_C(2N+h-1)) = 0$. But $\{h^1(\mathcal{O}_C(t))\}$ is a strictly decreasing sequence until it reaches 0, so $h^1(\mathcal{O}_C(2N+h-2)) = 0$. Now, we have an exact sequence

$$0 \to \mathcal{I}_X(2N+h-2) \to \mathcal{I}_{C'}(2N+h-2) \to \omega_C(2-2N-h) \to 0.$$

We have just seen that the third term in the associated long exact sequence is 0, and the first is 0 by construction. Therefore $h^0(\mathcal{I}_{C'}(2N+h-2)) = 0$ so of course $h^0(\mathcal{I}_{C'}(2N+h-3)) = 0$ as well. Therefore $C' \in \mathbf{L}_{nm}^{h-1}$ has maximal rank. $\blacksquare$

4. THE MAXIMAL RANK CASE

In this section we will make an intensive use of rank-2 reflexive sheaves on $\mathbb{P}^3$, and in particular of curvilinear sheaves (that is, sheaves with smooth sections). For the necessary background and results see [HA1, HA2, HA3, BO1, BO4].

We begin with a basic observation which we shall use repeatedly in this section.

PROPOSITION 4.1. (a) *Let G be a zero-dimensional subscheme of $\mathbb{P}^2$, and $(n_0, ..., n_{\sigma-1})$ its numerical character. Then*

$$A_{\sigma+k} = \sigma - A_\sigma - A_{\sigma+1} - \cdots - A_{\sigma+k-1} + h^1(\mathbb{P}^2, \mathcal{I}_G(\sigma+k))$$
$$- h^1(\mathbb{P}^2, \mathcal{I}_G(\sigma+k-1)) \qquad \text{for} \quad k \geq 0.$$

(b) *Two smooth maximal rank curves Y and Z of $\mathbf{L}_{mn}$ have the same cohomology (i.e., $h^i(\mathbb{P}^3, \mathcal{I}_Y(p)) = h^i(\mathbb{P}^3, \mathcal{I}_Z(p))$ and $h^i(Y, \mathcal{O}_Y(p)) = h^i(Z, \mathcal{O}_Z(p))$ for every i and p) if and only if they have the same numerical character.*

Proof. (a) This follows from the definition and Remark 1.9.

(b) The proof is the same as the proofs of 2.3 and 2.4 in [BO3]; it is only necessary to remember that for every n and for every plane H the multiplication induced by H, $\rho: H^1(\mathbb{P}^3, \mathcal{I}_Y(n)) \to H^1(\mathbb{P}^3, \mathcal{I}_Y(n+1))$, is trivial (same thing for Z). Then conclude by remembering that the numerical character of a set of points completely determines its first cohomology (part (a)) and hence also its zeroth cohomology. ∎

We want to give numerical conditions which turn out to be necessary and sufficient for the existence of a smooth maximal rank curve in $\mathbf{L}_{mn}$ with given numerical character. This result, together with [BM, GM], completes the classification of smooth maximal rank arithmetically Buchsbaum curves (Remark 4.8).

LEMMA 4.2. *Let $Y \in \mathbf{L}^h_{mn}$ be a smooth maximal rank curve with numerical character $N(Y) = (n_0, ..., n_{\sigma-1})$. Then*

$$\sigma = 2m + 2n + h - 1$$

$$A_\sigma = m - 1, \qquad \sum_{\tau \leq \sigma+2} A_\tau \geq n, \qquad A_{\sigma+1} \leq m + n + h.$$

If moreover $h^2(\mathbb{P}^3, \mathcal{I}_Y(\sigma)) = 0$, then $A_{\sigma+2}(Y) \geq n$.

Proof. From the definition of $\mathbf{L}^h_{mn}$ and the fact that Y has maximal rank we have

$$h^1(\mathbb{P}^3, \mathcal{I}_Y(2m+2n+h-2)) = m,$$
$$h^1(\mathbb{P}^3, \mathcal{I}_Y(2m+2n+h-1)) = n,$$

and

$$h^0(\mathbb{P}^3, \mathcal{I}_Y(2m+2n+h-1)) = 0.$$

Now, let $H = \mathbb{P}^2$ be a general plane. Our strategy will be to study the numbers $A_i(Y)$ by considering the hyperplane section of Y. From the usual exact sequence

$$0 \to \mathscr{I}_Y(-1) \to \mathscr{I}_Y \to \mathscr{I}_{Y\cap H} \to 0 \qquad (*)$$

we get

$$h^0(H, \mathscr{I}_{Y\cap H}(2m+2n+h-2)) = 0$$
$$h^0(H, \mathscr{I}_{Y\cap H}(2m+2n+h-1)) = m,$$

Since σ is the lowest degree of a curve in H containing $Y\cap H$, we have $\sigma = 2m+2n+n-1$.

Recall the following Riemann–Roch formula:

$$\chi(\mathscr{I}_Y(p)) = \tfrac{1}{6}(p+3)(p+2)(p+1) - pd - 1 + g.$$

Note that in our case

$$\chi(\mathscr{I}_Y(p)) = h^2(\mathbb{P}^3, \mathscr{I}_Y(p)) \qquad \text{if} \quad p \leqslant 2m+2n+h-3$$
$$\chi(\mathscr{I}_Y(p)) = h^2(\mathbb{P}^3, \mathscr{I}_Y(p)) - h^1(\mathbb{P}^3, \mathscr{I}_Y(p))$$
$$\text{if} \quad 2m+2n+h-2 \leqslant p \leqslant 2m+2n+h-1.$$

Hence from Proposition 4.1(a) and (*) we have

$$\begin{aligned} A_\sigma &= \sigma + h^1(H, \mathscr{I}_{Y\cap H}(\sigma)) - h^1(H, \mathscr{I}_{Y\cap H}(\sigma-1)) \\ &= 2m+2n+h-1 - h^2(\mathbb{P}^3, \mathscr{I}_Y(2m+2n+h-3)) \\ &\quad + 2h^2(\mathbb{P}^3, \mathscr{I}_Y(2m+2n+h-2)) \\ &\quad - h^2(\mathbb{P}^3, \mathscr{I}_Y(2m+2n+h-1)) + n - m \\ &= m+3n+h-1 - \chi(\mathscr{I}_Y(2m+2n+h-1)) \\ &\quad - n + 2\chi(\mathscr{I}_Y(2m+2n+h-2)) + 2m - \chi(\mathscr{I}_Y(2m+2n+h-3)) \\ &= m-1. \end{aligned}$$

Again using Proposition 4.1(a) we get

$$\begin{aligned} \sum_{\tau \leqslant \sigma+2} A_\tau &= \sigma - A_\sigma - A_{\sigma+1} = h^1(H, \mathscr{I}_{Y\cap H}(\sigma)) - h^1(H, \mathscr{I}_{Y\cap H}(\sigma+1)) \\ &= n + h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma-1)) - 2h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) + h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma+1)) \geqslant n \end{aligned}$$

since $h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma-1)) - h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) \geqslant h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) - h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma+1))$ (thanks to [BM], Lemma 3.4]). Moreover,

$$A_{\sigma-1} = \sigma - A_\sigma - \sum_{\tau \geqslant \sigma+2} A_\tau \leqslant 2m+2n+h-1-m+1 = m+n+h.$$

It remains to prove the last statement of the lemma. Suppose that $h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) = 0$. Since

$$\sigma - A_\sigma - A_{\sigma+1} = h^1(H, \mathscr{I}_{Y \cap H}(\sigma)) - h^1(H, \mathscr{I}_{Y \cap H}(\sigma+1))$$
$$\sigma - A_\sigma - A_{\sigma+1} - A_{\sigma+2} = h^1(H, \mathscr{I}_{Y \cap H}(\sigma+1)) - h^1(H, \mathscr{I}_{Y \cap H}(\sigma+2)),$$

we have

$$\begin{aligned} A_{\sigma+2} &= h^1(H, \mathscr{I}_{Y \cap H}(\sigma)) - 2h^1(H, \mathscr{I}_{Y \cap H}(\sigma+1)) + h^1(H, \mathscr{I}_{Y \cap H}(\sigma+2)) \\ &= n + h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma-1)) - 3h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) \\ &\quad + 3h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma+1)) - h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma+2)) \\ &= n + h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma-1)) \geqslant n. \quad \blacksquare \end{aligned}$$

This result allows to study the first shifts of $\mathbf{L}_{mn}$ in the maximal rank smooth case.

LEMMA 4.3. *In $\mathbf{L}^0_{mn}$ there is no smooth maximal rank curve. In $\mathbf{L}^1_{mn}$ the only possible numerical character for a smooth maximal rank curve is*

$$\begin{aligned} A_\sigma &= m-1 \\ A_{\sigma+1} &= m+n+1 \\ A_{\sigma+2} &= n \\ A_\tau &= 0 \qquad \textit{for every } \tau \geqslant \sigma+3 \ (\textit{remember that } \sigma = 2m+2n). \end{aligned} \tag{$*$}$$

Proof. In $\mathbf{L}^0_{mn}$ there is no smooth curve at all (every curve in $\mathbf{L}^0_{mn}$ must be reducible by Theorem 2.12; on the other hand, a Buchsbaum curve of this kind is connected—cf. [M3]).

Let Y be a smooth maximal rank curve in $\mathbf{L}^1_{mn}$, and let $N(Y) = (n_0, n_1, ..., n_{\sigma-1})$, $\sigma = 2m+2n$, be its numerical character. Since $h^0(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) = 0$ and $h^0(\mathbb{P}^3, \mathscr{I}_Y(\sigma+1)) \geqslant 2$ [GM, Corollary 4.3], there exist two irreducible surfaces of degree $2m+2n+1$ containing Y. The curve X linked to Y by these two surfaces is in $\mathbf{L}^0_{nm}$, since

$$h^1(\mathbb{P}^3, \mathscr{I}_Y(2m+2n-1)) = h^1(\mathbb{P}^3, \mathscr{I}_Y(2m+2n-1)) = m$$
$$h^1(\mathbb{P}^3, \mathscr{I}_Y(2m+2n-2)) = h^1(\mathbb{P}^3, \mathscr{I}_Y(2m+2n)) = n.$$

But the degree of a curve in $\mathbf{L}^0_{nm}$ is uniquely determined, and therefore

$$\begin{aligned} \deg(Y) &= (2m+2n+1)^2 - \deg(X) \\ &= (2m+2n+1)^2 - 2(m+n)^2 - 2m \\ &= 2(m+n)^2 + 2(m+n) + 2n + 1. \end{aligned}$$

A simple calculation shows that the degree of a curve with numerical character $(*)$ is exactly $2(m+n)^2+2(m+n)+2n+1$. Moreover, it is easy to observe that every other numerical character compatible with Lemma 4.2 must have $A'_\sigma = m-1$, $A'_{\sigma+1} = m+n-t$, $t \geqslant 0$, and therefore the corresponding degree must be strictly larger than $\deg(Y)$. Hence $(*)$ is the only possible numerical character for a smooth curve in $\mathbf{L}^1_{mn}$. ∎

With this preparation we are ready to give a set of conditions that the numerical character of a smooth maximal rank curve in $\mathbf{L}_{mn}$ must satisfy.

THEOREM 4.4. *Let Y be a smooth maximal rank curve in $\mathbf{L}^h_{mn}$, with numerical character $N(Y) = (n_0, n_1, ..., n_{\sigma-1})$, $\sigma = 2m+2n+h-1$. Then*

$$A_\sigma(Y) = m-1$$

$$A_{\sigma+1}(Y) \geqslant m+n+1$$

$$A_{\sigma+2}(Y) \geqslant n$$

$$A_{\sigma+3}(Y) \neq 0 \Rightarrow A_{\sigma+2}(Y) > n.$$

Proof. We have already seen that $A_\sigma(Y) = m-1$. For the rest, the proof is by induction on $h \geqslant 1$.

If $h=1$, we know that the only possible numerical character satisfies the thesis.

Now let Y be a smooth maximal rank curve in $\mathbf{L}^h_{mn}$, $h>1$, and as usual let $E(Y) = \#\{j \mid n_j = n_0\}$. We must have $\sigma = 2m+2n+h-1$, $n_0 \geqslant \sigma+2$ (thanks to Lemma 4.2) and $h^0(\mathbb{P}^3, \mathcal{I}_Y(\sigma)) = 0$. It is not hard (using Remark 1.9) to verify that

$$E(Y) = h^1(H, \mathcal{I}_{Y \cap H}(n_0-2))$$

$$0 = h^1(H, \mathcal{I}_{Y \cap H}(n_0-1)).$$

If $n_0 = \sigma+2$ and $E(Y) = n$, then we have $A_{\sigma+2}(Y) = n$, $A_{\sigma+3}(Y) = 0$, and therefore $A_{\sigma+1}(Y) = n+m+h$, and we are O.K. So we can suppose that either $n_0 > \sigma+2$, or $E(Y) > n$. In both cases we get $h^2(\mathbb{P}^3, \mathcal{I}_Y(n_0-3)) \neq 0$, $h^2(\mathbb{P}^3, \mathcal{I}_Y(n_0+t)) = 0$ for every $t \geqslant -2$ (from the usual exact sequence of restriction, recalling that the multiplication map $H^1(\mathbb{P}^3, \mathcal{I}_Y(v)) \to H^1(\mathbb{P}^3, \mathcal{I}_Y(v+1))$ induced by the linear form associated to H is trivial for every v).

Let us suppose n_0 even, $n_0 = 2p$ (the same arguments holds if n_0 is odd). We have

$$0 \neq h^2(\mathbb{P}^3, \mathcal{I}_Y(n_0-3)) = h^1(Y, \mathcal{O}_Y(2p-3))$$

$$= h^0(Y, \omega_Y(3-2p)) = h^0(Y, \omega_Y(5-2(p+1)).$$

Therefore in the exact sequence determined by a non-trivial section of $\omega_Y(5-2(p+1))$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-p-1) \to F \to \mathcal{I}_Y(p) \to 0. \qquad (^\circ)$$

F is a curvilinear sheaf with $c_1(F) = -1$, and we know that

$$h^1(\mathbb{P}^3, F(p-1)) = h^1(\mathbb{P}^3, \mathcal{I}_Y(2p-1)) = h^1(\mathbb{P}^3, \mathcal{I}_Y(n_0-1)) = 0$$
$$h^2(\mathbb{P}^3, F(p-2)) = h^2(\mathbb{P}^3, \mathcal{I}_Y(2p-2)) = h^2(\mathbb{P}^3, \mathcal{I}_Y(n_0-2)) = 0$$
$$h^3(\mathbb{P}^3, F(p-3)) = h^0(\mathbb{P}^3, F(-p)) = 0 \qquad \text{(see [HA2])}.$$

Hence $F(p)$ is globally generated (thanks to the Castelnuovo–Mumford theorem). Let Z be the smooth zeroset of a suitable general section of $F(p)$, with the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-p) \to F \to \mathcal{I}_Z(p-1) \to 0. \qquad (^{\circ\circ})$$

We first need some calculations in order to exhibit the relation between the numerical characters of Y and Z.

First of all, note that $Z \in \mathbf{L}_{mn}^{h-1}$, since

$$h^1(\mathbb{P}^3, \mathcal{I}_Z(2m+2n+h-3)) = m$$
$$h^1(\mathbb{P}^3, \mathcal{I}_Z(2m+2n+h-2)) = n;$$

moreover

$$\begin{aligned} h^0(\mathbb{P}^3, \mathcal{I}_Z(2m+2n+h-2)) &= h^0(\mathbb{P}^3, \mathcal{I}_Z(\sigma-1)) \\ &= h^0(\mathbb{P}^3, F(\sigma-p)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(\sigma-2p)) \\ &= h^0(\mathbb{P}^3, \mathcal{I}_Y(\sigma)) + h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(\sigma-2p-1)) = 0 \end{aligned}$$

since $2p-2 = n_0 - 2 \geqslant \sigma$, and therefore Z is a curve of maximal rank.

Let $N(Z) = (n_0(Z), n_1(Z), \ldots, n_{\tau-1}(Z))$ be its numerical character, where $\tau = \sigma - 1 = 2m+2n+h-2$; $N(Z)$ satisfies the thesis by induction.

Combining $(^\circ)$ and $(^{\circ\circ})$ we find for $k \geqslant -3$

$$\begin{aligned} h^2(\mathbb{P}^3, \mathcal{I}_Z(k) &= h^2(\mathbb{P}^3, F(k-p+1)) \\ &\quad + h^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k-2p+1)) - h^3(\mathbb{P}^3, F(k-p+1)) \\ &= h^2(\mathbb{P}^3, \mathcal{I}_Y(k+1)) \\ &\quad - h^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k-2p)) + h^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k-2p+1)) \\ &= h^2(\mathbb{P}^3, \mathcal{I}_Y(k+1)) \\ &\quad - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2p-k-4)) + h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2p-k-5)). \end{aligned}$$

Let us call

$$P(a) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a)) - 3h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a-1)) + 3h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a-2)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a-3)).$$

It is easy to verify that $P(a) = 0$ if $a < 0$, and $P(a) = 1$ if $a \geq 0$.

Having completed these calculations, let us begin.

$$\begin{aligned}
A_{\sigma+1}(Y) &= \sigma - A_\sigma(Y) + h^1(H, \mathcal{I}_{Y \cap H}(\sigma+1)) - h^1(H, \mathcal{I}_{Y \cap H}(\sigma)) \\
&= \sigma - (m-1) + 2h^2(\mathbb{P}^3, \mathcal{I}_Y(\sigma)) \\
&\quad - h^2(\mathbb{P}^3, \mathcal{I}_Y(\sigma+1)) - n - h^2(\mathbb{P}^3, \mathcal{I}_Y(\sigma-1)) \\
&= \sigma - A_\tau(Z) - n - h^2(\mathbb{P}^3, \mathcal{I}_Z(\sigma)) \\
&\quad + 2h^2(\mathbb{P}^3, \mathcal{I}_Z(\sigma-1)) - h^2(\mathbb{P}^3, \mathcal{I}_Z(\sigma-2)) - P(2p-\sigma-2) \\
&= \sigma - A_\tau(Z) + h^1(H, \mathcal{I}_{Z \cap H}(\sigma)) \\
&\quad - h^1(H, \mathcal{I}_{Z \cap H}(\sigma-1)) - P(2p-\sigma-2) \\
&= \sigma - 1 - A_\tau(Z) + h^1(H, \mathcal{I}_{Z \cap H}(\sigma)) - h^1(H, \mathcal{I}_{Z \cap H}(\sigma-1)) \\
&= \tau - A_\tau(Z) + h^1(H, \mathcal{I}_{Z \cap H}(\sigma)) - h^1(\mathbb{P}^3, \mathcal{I}_{Z \cap H}(\sigma-1)) \\
&= A_{\tau+1}(Z).
\end{aligned}$$

In this calculation we repeatedly used 4.1(a); we also used the fact that $2p - \sigma - 2 \geq 0$ and hence $P(2p - \sigma - 2) = 1$. Therefore

$$A_{\sigma+1}(Y) = A_{\tau+1}(Z) \geq m + n + 1$$

by induction. So we have proved the second assertion for Y.

Now let us examine $A_{\sigma+2}(Y)$. A similar calculation gives

$$\begin{aligned}
A_{\sigma+2}(Y) &= \sigma - A_\sigma(Y) - A_{\sigma+1}(Y) + h^1(H, \mathcal{I}_{Y \cap H}(\sigma+2)) \\
&\quad - h^1(H, \mathcal{I}_{Y \cap H}(\sigma+1)) \\
&= \sigma - A_\tau(Z) - A_{\tau+1}(Z) + 2h^2(\mathbb{P}^3, \mathcal{I}_Y(\sigma+1)) \\
&\quad - h^2(\mathbb{P}^3, \mathcal{I}_Y(\sigma+2)) - h^2(\mathbb{P}^3, \mathcal{I}_Y(\sigma)) \\
&= \sigma - 1 - A_\tau(Z) - A_{\tau+1}(Z) + h^1(H, \mathcal{I}_{Z \cap H}(\sigma+1)) \\
&\quad - h^1(H, \mathcal{I}_{Z \cap H}(\sigma)) + 1 - P(2p-\sigma-3) \\
&= A_{\tau+2}(Z) + 1 - P(2p-\sigma-3).
\end{aligned}$$

and in the same way

$$A_{\sigma+3}(Y) = A_{\tau+3}(Z) + P(2p-\sigma-3) - P(2p-\sigma-4).$$

We have several possibilities:

(a) $n_0 = \sigma + 2$. Hence $P(2p - \sigma - 3) = 0 = P(2p - \sigma - 4)$. Also,

$$0 = A_{\sigma+3}(Y) = A_{\tau+3}(Z),$$

and

$$A_{\sigma+2}(Y) = A_{\tau+2}(Z) + 1 > n.$$

(b) $n_0 > \sigma + 3$. In this case $P(2p - \sigma - 3) = 1 = P(2p - \sigma - 4)$ and hence

$$\begin{aligned} &A_{\sigma+3}(Y) = A_{\tau+3}(Z) \\ &A_{\sigma+2}(Y) = A_{\tau+2}(Z) \geqslant n \qquad \text{by induction.} \end{aligned}$$

Therefore

$$0 \neq A_{\sigma+3}(Y) \Rightarrow 0 \neq A_{\tau+3}(Z) \Rightarrow A_{\tau+2}(Z) > n \Rightarrow A_{\sigma+2}(Y) > n$$

again by induction.

(c) $n_0 = \sigma + 3$. In this case $P(2p - \sigma - 3) = 1$, $P(2p - \sigma - 4) = 0$, and

$$\begin{aligned} &A_{\sigma+3}(Y) = A_{\tau+3}(Z) + 1 \\ &A_{\sigma+2}(Y) = A_{\tau+2}(Z) \geqslant n. \end{aligned}$$

If $A_{\sigma+3}(Y) \geqslant 2$, then $A_{\tau+3}(Z) \neq 0$, and therefore $A_{\sigma+2}(Y) = A_{\tau+2}(Z) > n$ by induction.

So we have as a last possibility $n_0 = \sigma + 3$, $A_{\sigma+3}(Y) = 1$. But for such a curve, we have $h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) = 1$ and $h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma + 1)) = 0$. Therefore

$$h^1(H, \mathscr{I}_{Y \cap H}(\sigma + 2)) = 0, \qquad h^1(H, \mathscr{I}_{Y \cap H}(\sigma + 1)) = h^1(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) = 1.$$

Now,

$$h^1(H, \mathscr{I}_{Y \cap H}(\sigma)) = \sum_0^{\sigma-1} (n_i - \sigma - 1) = 2A_{\sigma+3}(Y) + A_{\sigma+2}(Y) = A_{\sigma+2}(Y) + 2,$$

and therefore

$$\begin{aligned} &h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma - 1)) \\ &\quad = h^1(H, \mathscr{I}_{Y \cap H}(\sigma)) + h^2(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) - n = h^1(H, \mathscr{I}_{Y \cap H}(\sigma)) + 1 - n \\ &\quad = A_{\sigma+2}(Y) + 3 - n. \end{aligned}$$

But now consider again the sequence (^{c}); since $H^3(\mathbb{P}^3, F(p-4))=0$ we get a surjective map $H^2(\mathbb{P}^3, \mathcal{I}_Y(2p-4)) \to H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-5)) \to 0$, and therefore

$$h^2(\mathbb{P}^3, \mathcal{I}_Y(\sigma-1)) = h^2(\mathbb{P}^3, \mathcal{I}_Y(2p-4)) \geqslant h^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-5)) = 4.$$

Hence

$$A_{\sigma+2}(Y)+3-n \geqslant 4, \qquad \text{and therefore } A_{\sigma+2}(Y) \geqslant n+1.$$

Thus we have also proved the third and the fourth assertions for Y.

If n_0 is odd it is enough to consider a curvilinear sheaf with $c_1=0$. ▮

COROLLARY 4.5. *Let $Y \in \mathbf{L}_{mn}$ be a smooth maximal rank curve, with numerical character $N(Y)=(n_0, n_1, ..., n_{\sigma-1})$. Then:*

$$N(Y) \text{ is without gaps}, \ \sigma \geqslant 2m+2n,$$

$$A_\sigma(Y) = m-1$$

$$A_{\sigma+1}(Y) \geqslant m+n+1$$

$$A_{\sigma+2}(Y) \geqslant n, \qquad A_{\sigma+2}(Y)=n \Rightarrow A_\rho(Y)=0 \text{ for every } \rho > \sigma+2.$$

Proof. The fact that $N(Y)$ is without gaps follows from [GP]. The other assertions follow from Lemma 4.2 and Theorem 4.4. ▮

Now we will see that these conditions are also sufficient. The technique is similar to the one used in [BM], with a more intensive use of liaison addition.

LEMMA 4.6. *For every $m, n, h \geqslant 1$ there exists a smooth maximal rank curve in $\mathbf{L}^h_{mn}$ with numerical character without gaps satisfying*

$$\sigma = 2m+2n+h-1$$

$$A_\sigma(Y) = m-1$$

$$A_{\sigma+1}(Y) = m+n+h$$

$$A_{\sigma+2}(Y) = n.$$

Proof. First of all notice that if such a curve exists, then its cohomology is given by the following diagram (using Proposition 4.1):

$\mathcal{I}_Y$		$\sigma-2$	$\sigma-1$	σ	$\sigma+1$	$\sigma+2$
h^0	...	0	0	0	*	*
h^1	...	0	m	n	0	0
h^2	...	$3n+h$	0	0	0	0

Note also that

$$\deg(Y) = 2(m+n)^2 + 2(m+n)h + \tfrac{1}{2}(h^2+h) + 2n, \qquad \sigma = 2m+2n+h-1.$$

We work by induction on h. First we construct the curve in $\mathbf{L}^1_{mn}$ by applying Schwartau's Liaison Addition procedure (Theorem 1.6) to a curve in $\mathbf{L}_m$ and a curve in $\mathbf{L}_n$, and then smoothing. The induction step is then done by smoothing a "basic double link" (with the terminology of [LR]). These smoothing techniques are developed in [BO4].

($h=1$) Let Z be a minimal smooth curve of maximal rank in $\mathbf{L}_n$(see [BM]); then

$$h^1(\mathbb{P}^3, \mathcal{I}_Z(2n-2)) = n, \qquad h^1(\mathbb{P}^3, \mathcal{I}_Z(t)) = 0 \quad \text{if} \quad t \neq 2n-2,$$
$$h^0(\mathbb{P}^3, \mathcal{I}_Z(2n-1)) = 0, \qquad h^0(\mathbb{P}^3, \mathcal{I}_Z(2n)) = 3n+1$$
$$h^2(\mathbb{P}^3, \mathcal{I}_Z(2n-4)) = 3n-1, \qquad h^2(\mathbb{P}^3, \mathcal{I}_Z(2n-2)) = 0.$$

Now let Y be a smooth maximal rank curve in $\mathbf{L}^1_m$ with numerical character $(2m+1, 2m+1, ..., 2m+1, 2m, ..., 2m)$, $A_{2m+1} = m+1$, $A_{2m} = m-1$ (see [BM, Theorem 5.3]). For such a curve we have

$$h^1(\mathbb{P}^3, \mathcal{I}_Y(2m-1)) = m, \qquad h^1(\mathbb{P}^3, \mathcal{I}_Y(t)) = 0 \quad \text{if} \quad t \neq 2m-1,$$
$$h^0(\mathbb{P}^3, \mathcal{I}_Y(2m)) = 0, \qquad h^0(\mathbb{P}^3, \mathcal{I}_Y(2m+1)) \neq 0$$
$$h^2(\mathbb{P}^3, \mathcal{I}_Y(2m-2)) \neq 0, \qquad h^2(\mathbb{P}^3, \mathcal{I}_Y(2m-1)) = 0.$$

Then it is possible to apply Proposition 3.1 of [BO4] and liaison addition to Z and Y by means of surfaces of degrees $2n$ and $2m+2$ in such a way that $Y \cup Z \cup (\Sigma_{2n} \cap \Sigma_{2m+2}) = C$ is smoothable with fixed cohomology. Let X be the smooth curve thus obtained. Notice that

$$h^1(\mathbb{P}^3, \mathcal{I}_X(2m+2n-1)) = h^1(\mathbb{P}^3, \mathcal{I}_Y(2m-1)) = m$$
$$h^1(\mathbb{P}^3, \mathcal{I}_X(2m+2n)) = h^1(\mathbb{P}^3, \mathcal{I}_Z(2n-2)) = n.$$

Also, the multiplication is trivial: in fact, the Hartshorne–Rao module of X is isomorphic to the Hartshorne–Rao module of C, which is a Buchsbaum curve by construction (both are isomorphic to the H^1-module of the same reflexive sheaf; see [BO4] for further details).

Hence $X \in \mathbf{L}^1_{mn}$. In order to prove that it has maximal rank it is enough to see that $h^0(\mathbb{P}^3, \mathcal{I}_X(2m+2n)) = 0$. But this follows immediately from the construction. From the uniqueness of the numerical character in the first shift of $\mathbf{L}_{mn}$ for maximal rank curves (Lemma 4.3) it follows that $X = X_1$ is the desired curve.

($h>1$) Suppose that we have X_{h-1}. Thanks to the discussion at the beginning of this proof we know that

$$h^1(\mathbb{P}^3, \mathscr{I}_{X_{h-1}}(\sigma(X_{h-1})+1)) = h^1(\mathbb{P}^3, \mathscr{I}_{X_{h-1}}(2m+2n+h-1)) = 0$$

$$h^2(\mathbb{P}^3, \mathscr{I}_{X_{h-1}}(\sigma(X_{h-1}))) = h^2(\mathbb{P}^3, \mathscr{I}_{X_{h-1}}(2m+2n+h-2)) = 0$$

$$h^2(\mathbb{P}^3, \mathscr{I}_{X_{h-1}}(\sigma(X_{h-1})-2)) = h^2(\mathbb{P}^3, \mathscr{I}_{X_{h-1}}(2m+2n+h-4)) \neq 0$$

and that X_{h-1} is contained in a smooth surface S of degree $\sigma(X_{h-1})+2 = 2m+2n+h$. Let H be a general plane; thanks to [BO4, Proposition 4.6] we know that $X_{h-1} \cup (S \cap H)$ is smoothable with fixed cohomology to a curve X. But it is simple to verify that X is in $\mathbf{L}_{mn}^h$ and it has maximal rank (since $h^0(\mathbb{P}^3, \mathscr{I}_X(2m+2n+h-1)) = 0$). Moreover

$$\begin{aligned}
\deg(X) &= \deg(X_{h-1}) + 2m + 2n + h \\
&= 2(m+n)^2 + 2(m+n)(h-1) + \tfrac{1}{2}(h-1)h + 2m + 2n + h + 2n \\
&= (2m+n)^2 + 2(m+n)h + \tfrac{1}{2}(h+1)h + 2n.
\end{aligned}$$

But the only numerical character in $\mathbf{L}_{mn}^h$ giving this degree is

$$A_\sigma = m-1, \qquad A_{\sigma+1} = m+n+h, \qquad A_{\sigma+2} = n, \tag{$*$}$$

since all the other allowed numerical characters (in the same shift) give degrees strictly larger (as in the proof of Lemma 4.3). Therefore the numerical character of X must be ($*$). This completes the proof. ∎

PROPOSITION 4.7. *Let $N = (n_0, n_1, \ldots, n_{\sigma-1})$ be a sequence of integers satisfying the following conditions*:

$$\sigma = 2m+2n+h-1, \qquad h \geq 1$$

N is without gaps

$$A_\sigma = m-1$$

$$A_{\sigma+1} \geq m+n+1$$

$$A_{\sigma+2} \geq n, \qquad A_{\sigma+3} \neq 0 \Rightarrow A_{\sigma+2} > n.$$

Then there exists a smooth maximal rank curve $X \in \mathbf{L}_{mn}^h$ whose numerical character is N.

Proof. Let us work by induction on $n \geq 1$.

If $n = 1$, there is only one possible character, and the construction of the corresponding curve was done in Lemma 4.7.

Now let us suppose $h > 1$.

(a) $A_{\sigma+2}=n$. Again, we did this job in Lemma 4.7, since in this case there is only one possible numerical character.

(b) $A_{\sigma+2}>n$. Suppose for instance that n_0 is odd, $n_0=2p+1$ (the same argument holds if n_0 is even). Consider the sequence of integers

$$N'=(n'_0, n'_1, ..., n'_{\tau-1}), \qquad \tau=\sigma-1,$$
$$n'_{\tau-1}=n_{\sigma-1}-1\geqslant\sigma-1=\tau;$$
$$n'_{\tau-2}=n_{\sigma-2}-1$$
$$\dots$$
$$n'_1=n_2-1$$
$$n'_0=n_1-1=\begin{cases}2p & \text{if } E>1\\ 2p-1 & \text{if } E=1.\end{cases}$$

Then, if we define A'_τ, $A'_{\tau+1}$ and so on as in Notation 1.10 (for N'), we have

$$A'_\tau=A_\sigma=m-1,$$
$$A'_{\tau+1}=A_{\sigma+1}\geqslant m+n+1$$

and either $n_0>\sigma+2$ (and this implies $A'_{\tau+2}=A_{\sigma+2}>n$) or $n_0=\sigma+2$ (and this implies $A'_{\tau+3}=A_{\sigma+3}=0$, $A'_{\tau+2}=A_{\sigma+2}-1\geqslant n$).

Hence N' satisfies the hypothesis for $h-1$, and so by induction there exists a smooth maximal rank curve $T\in\mathbf{L}^{h-1}_{mn}$ such that $N(T)=N^1$.

Now we try for a non-trivial section of $\omega_T(4-2p)$. Note that $h^0(T,\omega_T(4-2p))=h^2(\mathbb{P}^3,\mathscr{I}_T(2p-4))$, so it is enough to prove that $H^2(\mathbb{P}^3,\mathscr{I}_T(2p-4))\neq 0$.

We have a long exact sequence (where H is a general plane)

$$\dots 0\to H^1(\mathbb{P}^3,\mathscr{I}_T(2p-3))\to H^1(H,\mathscr{I}_{T\cap H}(2p-3))\to H^2(\mathbb{P}^3,\mathscr{I}_T(2p-4)\dots$$

(Recall that T is arithmetically Buchsbaum.) There are three possibilities:

(a) $2p-3>\tau$. It is enough to prove that $h^1(H,\mathscr{I}_{T\cap H}(2p-3))\neq 0$, since $H^1(\mathbb{P}^3,\mathscr{I}_T(2p-3))=0$.

(b) $2p-2=\tau$. It is enough to prove that $h^1(H,\mathscr{I}_{T\cap H}(2p-3))>n$.

(c) $2p-3=\tau-1$. It is enough to prove that $h^1(H,\mathscr{I}_{T\cap H}(2p-3))>m$.

Now,

(a) The condition $2p-3>\tau$ means $\sigma<2p-2$. In this case

$h^1(H, \mathscr{I}_{T\cap H}(2p-3)) = \sum_0^{\tau-1} (n_i(T)-2p+2)_+$ is bigger than zero since $n_0(T) \geq 2p-1$.

(b) In this case $A_{\sigma+3} \neq 0$, and hence $A_{\tau+2} = A_{\sigma+2} > n$ by hypothesis. So $h^1(H, \mathscr{I}_{T\cap H}(2p-3)) = h^1(H, \mathscr{I}_{T\cap H}(\tau)) = \sum_0^{\tau-1} (n_i(T)-\tau-1)_+ > n$.

(c) In this case $n_0 = \sigma+2$. Therefore

$$E = A_{n_0} = A_{\sigma+2} > n$$
$$A_{\tau+2}(T) = A_{\sigma+2} - 1 \geq n$$
$$A_{\tau+1}(T) \geq m+n+1.$$

Hence

$$h^1(H, \mathscr{I}_{T\cap H}(2p-3)) = h^1(\mathbb{P}^3, \mathscr{I}_{T\cap H}(\tau-1)) = \sum_0^{\tau-1} (n_i(T)-\tau)_+$$
$$\geq 2n+m+n+1 = 3n+m+1.$$

Therefore in every case $h^0(T, \omega_T(4-2p)) \neq 0$, and we can write the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-p) \to F \to \mathscr{I}_T(p) \to 0,$$

where F is a curvilinear sheaf with $c_1 = 0$ and

$$h^1(\mathbb{P}^3, F(t-p-1)) = h^1(\mathbb{P}^3, \mathscr{I}_T(\tau-1)) = m$$
$$h^1(\mathbb{P}^3, F(t-p)) = h^1(\mathbb{P}^3, \mathscr{I}_T(\tau)) = n$$
$$h^1(\mathbb{P}^3, F(s)) = 0 \qquad \text{if} \quad s \geq \tau-p+1$$
$$h^2(\mathbb{P}^3, F(p-1)) = 0$$

since $h^1(H, \mathscr{I}_{T\cap H}(2p)) = \sum_0^{\tau-1} (n_i(T)-2p-1)_+ = 0 \quad (n_i(T) \leq 2p)$, and therefore $h^2(\mathbb{P}^3, \mathscr{I}_T(2p-1)) = 0$.

Note that $H^1(\mathbb{P}^3, F(p)) = 0$, since $\tau \leq 2p-1 \Rightarrow p \geq \tau-p+1$. Therefore $F(p+1)$ is globally generated and it has a section with smooth zeroset X, and there is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-p-1) \to F \to \mathscr{I}_X(p+1) \to 0.$$

The relation between the numerical characters of X and T can be computed as in Theorem 4.4. This direct calculation shows that X is a smooth maximal rank curve in $\mathbf{L}^h_{mn}$ with the announced numerical character N. ∎

Remark 4.8. As we mentioned above, these results complete the cohomological classification of the smooth maximal rank Buchsbaum curves. In fact, thanks to [GM], the only Buchsbaum classes containing maximal rank curves are

(a) the class of arithmetically Cohen–Macaulay curves (studied in [GP]);

(b) the $\mathbf{L}_m$'s (studied in [BM]);

(c) the $\mathbf{L}_{mn}$'s.

5. An Example

Two natural questions in studying liaison and Buchsbaum curves are the following (roughly stated) ones:

(1) Does there exist a Buchsbaum curve which is a deformation of an irreducible family of non-Buchsbaum curves?

(2) Is it possible to find an irreducible family of curves such that the Hartshorne–Rao module does not change in the dimensions of its components but changes in its multiplicative structure?

These questions are closely related to each other, and as an application of previous results we will give an answer to both with the same example (in Italian, "prendere due piccioni con una fava").

We recall how we constructed a minimal, reducible curve in $\mathbf{L}_{11}$ (Proposition 2.1). We took two general pairs of skew lines, $L_1 \cup L_2$ and $L_3 \cup L_4$, a quadric Σ_2 containing the first pair, and a cubic surface Σ_3 containing the second pair. Let us call $Y = \Sigma_3 \cap \Sigma_2$. Our curve was $C = \bigcup_{i=1}^4 L_i \cup Y$. In fact, we proved that $C \in \mathbf{L}_{11}^0$, $\deg(C) = 10$, $p_a(C) = 10$, and that the cohomology of C is

$\mathscr{I}_C$	0	1	2	3	4
h^0	0	0	0	0	1
h^1	0	0	1	1	0
h^2	*	*	0	0	0

If things are chosen generally, C is smoothable: since $H^1(Y, \mathscr{N}_Y) = 0$, it is enough to apply Corollary 4.2 and Remark 4.2.2 of [HH].

By semicontinuity (from the cohomology of $\mathscr{I}_C$ above and the fact that $\chi = h^0 - h^1 + h^2$ is preserved in each degree), the general smooth curve X has dimensionally the same cohomology of Y, i.e., $h^i(\mathbb{P}^3, \mathscr{I}_X(t)) = h^i(\mathbb{P}^3, \mathscr{I}_C(t))$ for every i and t. So, if X is Buchsbaum, it must belong to $\mathbf{L}_{11}^0$, exactly as C. But in $\mathbf{L}_{11}^0$ there is no smooth curve: this was proved in Section 2.

Hence the general curve X is not Buchsbaum and its Hartshorne–Rao module is dimensionally equivalent to that of C but has a different multiplicative structure.

6. Questions and Comments

The technique introduced in Section 2 of using Theorem 1.7 to produce a "good" link can be used in non-Buchsbaum cases, but is not always useful. There are cases in which it is equally useful, however.

An example where it is helpful is a liaison class corresponding to a Hartshorne–Rao module M which is 1-dimensional in each of t consecutive degrees, and the multiplication between any two consecutive components is non-trivial for general $L \in S_1$ and trivial for a fixed hyperplane of S_1. An example of a curve with this module is the disjoint union C of a line and a plane curve of degree t, in which case the leftmost non-zero component is in degree 0 (cf. [M1, Example 2.3]). By [M1, Proposition 2.8], this is the leftmost possible shift for a curve in this liaison class, and clearly the smallest degree of a surface containing a curve in this liaison class is two (the union of two planes). In fact, there exist linear forms which annihilate M but do not vanish on any component of the minimal curve, and this curve acts very much as though it were Buchsbaum. (See also [GM].) Then most of the techniques and results of this paper extend directly to this case with $2N$ replaced by 2.

On the other hand, an example of a liaison class for which this does not help is that of a double line in $\mathbb{P}^3$ of arithmetic genus $\leqslant -2$ (cf. [M2]). Here the double lines are again minimal–one can in fact check that $e(C) = -2$, so they actually satisfy the hypothesis of [LR]. However, the rightmost non-zero component of the Hartshorne–Rao module of a double line occurs in degree $-1-g$ ($g =$ genus). Hence, as we have done in Section 2, we can only guarantee a link using surfaces of degree 2 and $-g+1$, while there actually exists a link using two surfaces of degree 2. Note, though, that the Hartshorne–Rao module here is again annihilated by certain linear forms! Possibly the key difference between this and the example in the previous paragraph is that here the linear forms which annihilate the module *all* vanish on a component of *every* minimal curve in the liaison class.

Another important question is the following one: How far is the example of Section 5 generalizable? When is a Buchsbaum curve a specialization of non-Buchsbaum curves (maybe singular)? This would involve a detailed study of "deformation of linkages."

Moreover, now the natural question is: How is it possible to bring this kind of classification result to non-Buchsbaum classes? It seems natural to relate to the results of one of the authors (see [M1]).

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